Note: In this problem set, expressions in green cells match corresponding expressions in the text answers.

1 - 6 Euler for systems and second-order ODEs
Solve by the Euler's method. Graph the solution in the $y_1 y_2$ – plane. Calculate the errors.

1. $y_1'[x] = 2 y_1[x] - 4 y_2[x]$,
$y_2'[x] == y_1[x] - 3 y_2[x]$, $y_1[0] == 3$, $y_2[0] == 0$

```
Clear["Global`*"]
```

If both functions that I use have the capability to solve a system of ODEs, that seems remarkable. The exact case is with

```
s1 = DSolve[{y1'[x] == 2 y1[x] - 4 y2[x],
   y2'[x] == y1[x] - 3 y2[x], y1[0] == 3, y2[0] == 0}, {y1, y2}, x]
```

$\{\{y1 \to \text{Function}[\{x\}, e^{-2x}(-1 + 4 e^{3x})],$
$y2 \to \text{Function}[\{x\}, e^{-2x}(-1 + e^{3x})]\}\}$

```
p1 = Plot[{y1[x] /. s1, y2[x] /. s1},
   {x, -1, 4}, PlotStyle → {{Blue, Thickness[0.008]},
     {RGBColor[0.8, 0.3, 0.2], Thickness[0.008]}}];
```

It's particularly satisfying to see that the options which enhance accuracy and precision work as well for the dual equations as for the individual ones.

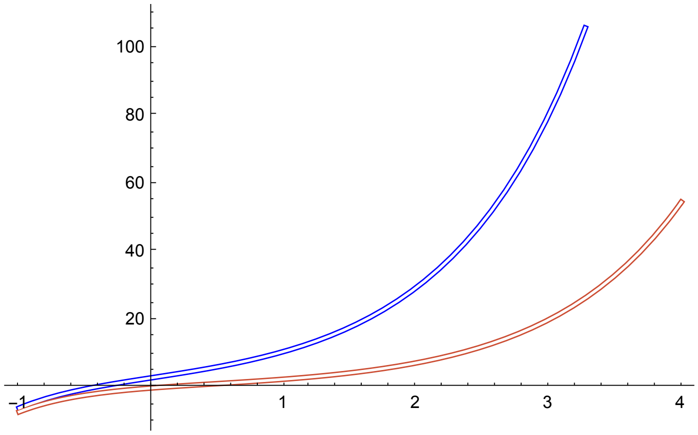
```
s2 = NDSolve[{y1'[x] == 2 y1[x] - 4 y2[x],
   y2'[x] == y1[x] - 3 y2[x], y1[0] == 3, y2[0] == 0}, {y1, y2}, {x, -1, 4},
  AccuracyGoal → 16, PrecisionGoal → 16, WorkingPrecision → 20]
```

```
{{y1 → InterpolatingFunction[ Domain: {{-1.0000000000000000000, 4.0000000000000000000}}
                              Output: scalar ],
  y2 → InterpolatingFunction[ Domain: {{-1.0000000000000000000, 4.0000000000000000000}}
                              Output: scalar ]}}
```

```
p2 = Plot[{y1[x] /. s2, y2[x] /. s2}, {x, -1, 4},
   PlotStyle → {{White, Thickness[0.004]}, {White, Thickness[0.004]}}];
```

The plots work exactly like one of the previous problems in section 21.1 or 21.2, only in duplicate.

```
Show[p1, p2]
```



The values table can easily be adjusted to accommodate four functions. Everything adjusts and the accuracy between pairs of table columns seems to be 9S or better.

```
TableForm[Table[
  NumberForm[{y1[x] /. s1, y1[x] /. s2, y2[x] /. s1, y2[x] /. s2}, {8, 8}],
  {x, -1, 4, 0.4}]]
```

```
{{-5.91753830}, {-5.91753830}, {-7.02117670}, {-7.02117670}}
{{-1.12487040}, {-1.12487030}, {-2.77130530}, {-2.77130520}}
{{1.78309830}, {1.78309830}, {-0.67309394}, {-0.67309394}}
{{4.21529100}, {4.21529100}, {0.55108271}, {0.55108271}}
{{6.98728100}, {6.98728100}, {1.52092460}, {1.52092460}}
{{10.73779200}, {10.73779200}, {2.58294650}, {2.58294650}}
{{16.15999000}, {16.15999000}, {3.99438990}, {3.99438990}}
{{24.17126600}, {24.17126600}, {6.02232370}, {6.02232370}}
{{36.08777700}, {36.08777600}, {9.01273620}, {9.01273610}}
{{53.84943600}, {53.84943500}, {13.45822100}, {13.45822100}}
{{80.33966900}, {80.33966800}, {20.08305800}, {20.08305800}}
{{119.85529000}, {119.85529000}, {29.96298600}, {29.96298600}}
{{178.80424000}, {178.80424000}, {44.70068400}, {44.70068400}}
```

3. $y''[x] + \frac{1}{4} y[x] == 0,\ y[0] == 1,\ y'[0] == 0$

```
Clear["Global`*"]
```

This problem is not a system but rather a simple second order ODE.

$$s1 = \text{DSolve}\left[\left\{y''[x] + \frac{1}{4} y[x] == 0,\ y[0] == 1,\ y'[0] == 0\right\}, y, x\right]$$

$$\left\{\left\{y \to \text{Function}\left[\{x\}, \text{Cos}\left[\frac{x}{2}\right]\right]\right\}\right\}$$

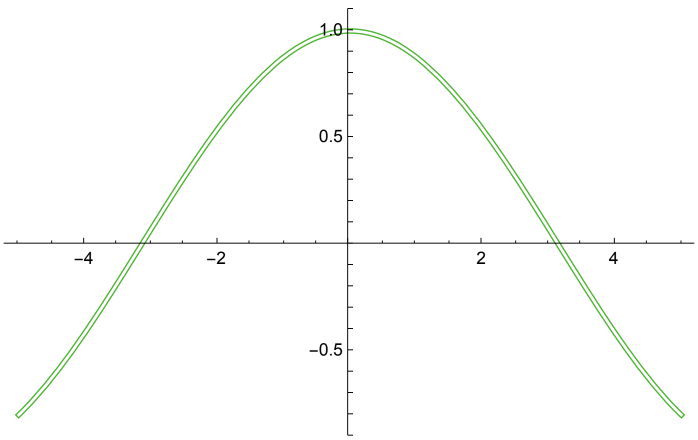
```
p1 = Plot[y[x] /. s1, {x, -5, 5},
   PlotStyle → {RGBColor[0.3, 0.7, 0.2], Thickness[0.008]}];
```

```
s2 = NDSolve[{y''[x] + 1/4 y[x] == 0, y[0] == 1, y'[0] == 0}, y, {x, -5, 5},
  AccuracyGoal → 16, PrecisionGoal → 16, WorkingPrecision → 20]
```

```
{{y → InterpolatingFunction[ Domain: {{-5.0000000000000000000, 5.0000000000000000000}}
                             Output: scalar ]}}
```

The plot is delivered as expected. Once the equation is solved, everything that follows is exactly like first order.

```
p2 = Plot[{y[x] /. s2}, {x, -5, 5}, PlotStyle → {White, Thickness[0.004]}];
Show[p1, p2]
```



The table of comparison values between functions has the close equivalence of a number of the recent problem solutions.

```
TableForm[
 Table[NumberForm[{y[x] /. s1, y[x] /. s2}, {8, 8}], {x, -5, 5, 0.8}]]
```

```
{{-0.80114362}, {-0.80114362}}
{{-0.50484610}, {-0.50484610}}
{{-0.12884449}, {-0.12884449}}
{{0.26749883}, {0.26749883}}
{{0.62160997}, {0.62160997}}
{{0.87758256}, {0.87758256}}
{{0.99500417}, {0.99500417}}
{{0.95533649}, {0.95533649}}
{{0.76484219}, {0.76484219}}
{{0.45359612}, {0.45359612}}
{{0.07073720}, {0.07073720}}
{{-0.32328957}, {-0.32328957}}
{{-0.66627602}, {-0.66627602}}
```

5. y''[x] - y[x] == x, y[0] == 1, y'[0] == -2

A problem consisting of another second order ODE.

```
Clear["Global`*"]
```

Mathematica can solve this one in closed form

```
s1 = DSolve[{y''[x] - y[x] == x, y[0] == 1, y'[0] == -2}, y, x]
```

```
{{y → Function[{x}, -e^-x (-1 + e^x x)]}}
```

```
p1 = Plot[y[x] /. s1, {x, -5, 5},
   PlotStyle → {RGBColor[0.7, 0.2, 0.7], Thickness[0.008]}];
```

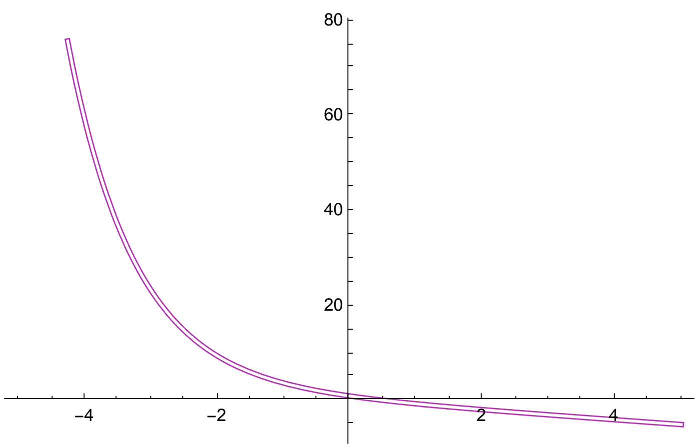
as well as in numerical approximation.

```
s2 = NDSolve[{y''[x] - y[x] == x, y[0] == 1, y'[0] == -2}, y, {x, -5, 5},
  AccuracyGoal → 16, PrecisionGoal → 16, WorkingPrecision → 20]
```

```
{{y → InterpolatingFunction[ Domain: {{-5.0000000000000000000, 5.0000000000000000000}}
                             Output: scalar ]}}
```

```
p2 = Plot[{y[x] /. s2}, {x, -5, 5}, PlotStyle → {White, Thickness[0.004]}];
```

```
Show[p1, p2]
```



And modified with suitable enhancers, the table comparison shows very good correlation between values of the two functions.

```
TableForm[
 Table[NumberForm[{y[x] /. s1, y[x] /. s2}, {8, 8}], {x, -5, 5, 0.8}]]
{{153.41316000}, {153.41316000}}
{{70.88633100}, {70.88633100}}
{{33.36410000}, {33.36410000}}
{{16.06373800}, {16.06373800}}
{{7.84964750}, {7.84964750}}
{{3.71828180}, {3.71828180}}
{{1.42140280}, {1.42140280}}
{{-0.05118836}, {-0.05118836}}
{{-1.15340300}, {-1.15340300}}
{{-2.08919680}, {-2.08919680}}
{{-2.95021290}, {-2.95021290}}
{{-3.77762920}, {-3.77762920}}
{{-4.58994820}, {-4.58994820}}
```

7 - 10 RK for systems
Solve by the classical RK

7. The ODE in problem 5. By what factor did the error decrease?

The error is so miniscule that in my opinion there is no need to try to get it to decrease.

9. The system in problem 1.

The system in problem 1 has been solved definitively, I believe.

11. Pendulum equation
y”[x] + Sin[y[x]] == 0, y[π] == 0, y’[π] == 1, as a system, 20 steps. How does your result fit into figure 93 in section 4.5?

```
Clear["Global`*"]
```

I think I see how this could be treated as a system, with sine handled as a separate function. But it was easier to try to solve it monolithically first, and it worked. The text answer page does not show a result for the solution function y[x]. I am only assuming that Mathematica got it right.

```
eqn = {y''[x] + Sin[y[x]] == 0, y[π] == 0, y'[π] == 1}
{Sin[y[x]] + y''[x] == 0, y[π] == 0, y'[π] == 1}
```

```
s1 = Simplify[DSolve[{y''[x] + Sin[y[x]] == 0, y[π] == 0, y'[π] == 1}, y, x]]
```

$$\{\{y \to \text{Function}[\{x\}, 2\,\text{JacobiAmplitude}[\tfrac{1}{2}(-\pi + x), 4]]\}\}$$

I’m not sure if the following constitutes a successful check of the **DSolve** activities, but it may.

```
eqn /. s1
```

$$\left\{\left\{-2\,\text{JacobiCN}\left[\frac{1}{2}(-\pi+x),4\right]\text{JacobiSN}\left[\frac{1}{2}(-\pi+x),4\right]+\text{Sin}\left[2\,\text{JacobiAmplitude}\left[\frac{1}{2}(-\pi+x),4\right]\right]=0,\ \text{True},\ \text{True}\right\}\right\}$$

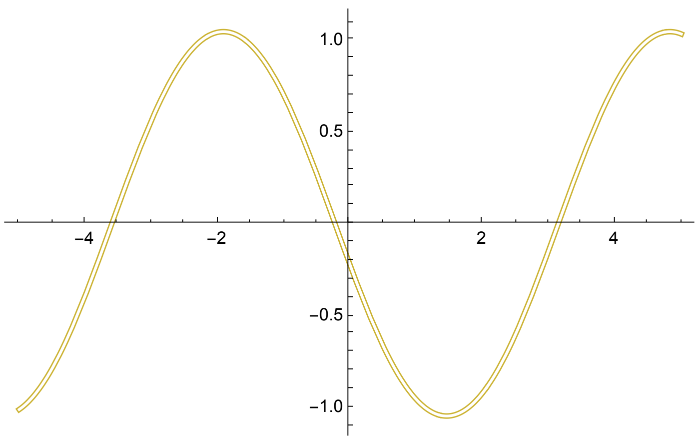
```
p1 = Plot[y[x] /. s1, {x, -5, 5},
    PlotStyle → {RGBColor[0.8, 0.7, 0.2], Thickness[0.008]}];
```

```
s2 = NDSolve[{y''[x] + Sin[y[x]] == 0, y[π] == 0, y'[π] == 1}, y, {x, -5, 5},
   AccuracyGoal → 16, PrecisionGoal → 16, WorkingPrecision → 20]
```

```
{{y → InterpolatingFunction[ Domain: {{-5.0000000000000000000, 5.0000000000000000000}}
                             Output: scalar ]}}
```

```
p2 = Plot[{y[x] /. s2}, {x, -5, 5}, PlotStyle → {White, Thickness[0.004]}];
```

```
Show[p1, p2]
```



The **Simplify** command I used in the definition of s1 did not do as I had hoped. It did not eliminate the phantom imaginary Arg from the function values of s1. These would show up in the table without the exclusionary Re command on s1. I feel fairly confident in stripping Arg, since all the Arg values are far less than default **Chop**. The comparison of the remaining real parts shows an agreeably close equivalence.

```
TableForm[
 Table[NumberForm[{Re[y[x]] /. s1, y[x] /. s2}, {8, 8}], {x, -5, 5, 0.8}]]
{{-1.01161520}, {-1.01161520}}
{{-0.56405817}, {-0.56405817}}
{{0.20005255}, {0.20005255}}
{{0.84859892}, {0.84859892}}
{{1.04140840}, {1.04140840}}
{{0.69811386}, {0.69811386}}
{{-0.02990360}, {-0.02990360}}
{{-0.74057792}, {-0.74057792}}
{{-1.04584740}, {-1.04584740}}
{{-0.81332394}, {-0.81332394}}
{{-0.14112048}, {-0.14112048}}
{{0.61277079}, {0.61277079}}
{{1.02486590}, {1.02486590}}
```